PD2 • Learning from mistakes and misconceptions

Purpose

To encourage participants to:

- reflect on the nature and causes of learners' mistakes and misconceptions;
- consider ways in which we might use these mistakes and misconceptions constructively to promote learning.

Materials required

For each participant you will need:

- Sheet PD2.1 – *Analysing learners' work*;
- Sheet PD2.2 – *Commentary on learners' work*;
- Sheet PD2.3 – *Generalisations made by learners*;
- Sheet PD2.4 – *Some principles to consider*;
- Sheet PD2.5 – *Looking at one session from the resource*;
- at least one of the following sessions from the resource:

  N1 Ordering fractions and decimals
  N2 Evaluating statements about number operations
  A4 Evaluating algebraic expressions
  SS4 Evaluating statements about length and area
  S2 Evaluating probability statements

Supporting materials

To support this session, you may wish to use:

- extract from the DVD-ROM in *Thinking about learning/Using misconceptions/Example*;
- extract from DVD-ROM in *Planning learning/Session 2*;
- PowerPoint presentation in *Materials/Professional development* on the DVD-ROM. This will be useful when running the sessions and includes slides of the aims, and of appropriate handouts and tasks.

Time needed

From 1 to 2 hours.

Note
The learners' work and the commentaries used in this section are taken from Higgins S., Ryan J., Swan M. and Williams J., *Learning from mistakes, misunderstandings and misconceptions in mathematics*, in Thompson I. (ed.), *National Numeracy and Key Stage 3 Strategies* (DfES 0527/2002 edn), London, 2002, DfES.

Suggested activities

1. Assessing learners' responses to diagnostic questions

Give each participant a copy of Sheet PD2.1 – *Analysing learners' work* and ask them to read the (genuine) examples shown on pages PD2-6 to PD2-9. Each page contains the work of one learner. Ask participants to work in pairs and use the grid in Sheet PD2.1 to write a few lines summarising:

- the nature of the errors that have been made by each learner;
- the thinking that may have led to these errors.

Ask each pair to discuss their ideas with the whole group. The PowerPoint presentation may be used to facilitate this discussion.

You may like to give each participant a copy of Sheet PD2.2 – *Commentary on learners' work* and discuss how it compares with their own conclusions.

2. Why do learners make mistakes?

Through discussion, draw out from participants the different possible causes of the mistakes that learners make. These may be due to lapses in concentration, hasty reasoning, memory overload or a failure to notice important features of a problem. Other mistakes, however, may indicate alternative ways of reasoning. Such 'misconceptions' should not be dismissed as 'wrong thinking'; they may be **necessary** stages of conceptual development.

Give out copies of Sheet PD2.3 – *Generalisations made by learners* and consider the errors and underlying generalisations in each statement. Ask participants for examples to add to the list.

Such statements are valid in many contexts that occur earlier in a learner's education, when they work in limited contexts that do not generalise. For example, when children deal solely with natural numbers, they infer that 'when you multiply by ten you just add a nought'. Later on, this leads to errors such as $3.4 \times 10 = 3.40$
Many 'misconceptions' in learners' work may be attributed to the use of such **local generalisations**.

As a group, discuss the following questions.

- Can you think of other generalisations that are only true for limited contexts?
- For what contexts do the following generalisations work? In what contexts are they invalid?
  - If I subtract something from 12, the answer will be smaller than 12.

– The square root of a number is smaller than the number.
– All numbers can be written as proper or improper fractions.
– The order in which you multiply does not matter.
– You can differentiate any function.
– You can integrate any function.

3. What do we do with mistakes and misconceptions?

There are two common ways of reacting to learners' errors and misconceptions:

- Avoid them whenever possible.

  'If I warn learners about the misconceptions as I teach, they are less likely to happen. Prevention is better than cure.'

- Use them as learning opportunities.

  'I actively encourage learners to make mistakes and to learn from them.'

Discuss these contrasting views with the group.

Now give each participant a copy of Sheet PD2.4 – *Some principles to consider*. How do participants feel about this advice?

4. How are mistakes and misconceptions addressed in the sessions?

Give each participant Sheet PD2.5 – *Looking at one session from the resource* and a copy of at least one of the following sessions:

N1 Ordering fractions and decimals
N2 Evaluating statements about number operations
A4 Evaluating algebraic expressions
SS4 Evaluating statements about length and area
S2 Evaluating probability statements

There is a film sequence of a session planned round S2 and S3 in *Thinking about learning/Using misconceptions* on the DVD-ROM. This could be viewed by participants completing Sheet PD2.4 either as a group or individually, after the session. A more detailed look at this whole session can be found in *Planning learning/Session 2* on the DVD-ROM.

Ask participants to work in pairs to complete the sheet, with reference to the session. Pairs who finish quickly can be asked to consider another session.

Ask participants to consider the learning points that might be drawn out during a whole group discussion with learners after each activity in the session.

BLANK FOR NOTES

Sheet PD2.1 – *Analysing learners' work*

Look at the (genuine) examples of learners' work shown on the following pages.
Each page contains the work of one learner.

For each learner, write a few lines in the grid summarising:

- the nature of the errors that have been made;
- the thinking that may have led to these errors.

Compare what you have written with others in your group.

Learner's name and topic	**Errors made and the thinking that may have led to these errors**
Saira Fractions and decimals	
Damien Multiplication and division	
Julia Perimeter and area	
Jasbinder Algebraic notation	

Sheet PD2.1 – *Analysing learners' work* (continued)

Saira: Fractions and decimals

4. Write the following fractions as **decimals**:

a) six tenths ~~6l10~~ 06.10

b) four fifths ~~4 5~~ ~~54~~ 04.50

c) sixteen tenths ~~1.60~~ 16.10

d) three hundredths ~~0.100~~ ~~3.10~~ 30.10

---

11. Write these numbers in order of size, from **smallest** to **largest**:

0.625 0.25 0.3753 0.125 0.5

0.3753, 0.625, 0.125, 0.25, 0.5,

Smallest **Largest**

Explain how you chose the smallest.

I chose 0.3753 as the smallest as it is the one which is furthest away from 0.5 which is 0.5 away from being a whole number

---

12. Write these fractions in order of size, from smallest to largest:

$\frac{1}{2}$ $\frac{3}{8}$ $\frac{5}{16}$ $\frac{5}{8}$ $\frac{1}{4}$

5/16, 3/8, 5/8, 1/4, 1/2,

Smallest **Largest**

Explain how you chose the smallest.

because 5/16 is much smaller than 1/2 if you cut the fraction out of a cake there would be a lot of 16 small pieces instead of 2 large pieces

Sheet PD2.1 – *Analysing learners' work* (continued)

Damien: Multiplication and division

1. Do these in your head and write down your answers as decimals:

(a) $4 \div 20 =$ 5.0

(b) $6 \times 0.5 =$ 3.0

(c) $10 \div 0.5 =$ 2

(d) $0.7 \div 0.7 =$ 0.7

(e) $0.2 \times 0.4 =$ 0.8

(f) $60 \div 0.3 =$ 20

(g) $60 \times 0.3 =$ 18.0

(h) $16 \div 20 =$ ....

5. The answer to $26.12 \div 0.286$ will be....
Ring two correct statements

Bigger than 26 | Smaller than 26

Bigger than 13 | Smaller than 13

Give a **rough estimate** of the answer: 77.00

7.
(a) The boxes contain **six** statements.
Tick every statement that means the same as **85 ÷ 17**

How many 17's go into 85 ? ✓ | What fraction of 85 is 17 ?

$85\overline{)17}$ ✓ | $17\overline{)85}$ | $\frac{17}{85}$ ✓ | $\frac{85}{17}$

(b) **Tick every** statement that means the same as **19 ÷ 76**

How many 19's go into 76? ✓ | What fraction of 76 is 19 ?

$76\overline{)19}$ | $19\overline{)76}$ ✓ | $\frac{19}{76}$ ✓ | $\frac{76}{19}$

Sheet PD2.1 – *Analysing learners' work* (continued)

Julia: Perimeter and area

3. Explain, in your own words, the meanings of the terms:

Perimeter Is the outside of a shape

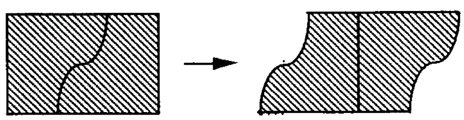
Area The flat surface of a shape

Volume The length + surface area eg the whole shape.

eg Inside of shape

4. You cut rectangle A, and arrange the pieces to make a new shape B, like this:

A B



Ring two statements that are true:

The area of A is greater than the area of B

The area of A is less than the area of B

Both areas are the same

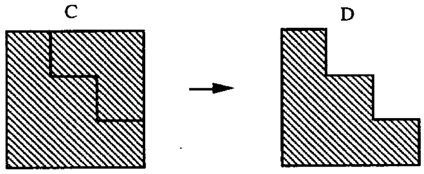
The perimeter of A is greater than the perimeter of B

The perimeter of A is less than the perimeter of B

Both perimeters are the same

5. You cut a piece out of C and throw the piece away. You are left with piece D:

C D



Ring two statements that are true:

The area of C is greater than the area of D

The area of C is less than the area of D

Both areas are the same

The perimeter of C is greater than the perimeter of D

The perimeter of C is less than the perimeter of D

Both perimeters are the same

Sheet PD2.1 – *Analysing learners' work* (continued)

Jasbinder: Algebraic notation

6 A piece of rope 5 metres long is cut into two pieces.
One piece is x metres long.
How long is the other piece? ......2.5 metres......

There are 24 hours in one day.
How many hours are there in y days? ...~~y x 24~~ = ~~24 hours~~... y = 3 = 72 hours.

It costs £140 to hire a coach.
This cost is shared equally among n people.
How much does each person pay? ......n = 14 ... = £10 each...

A plumber charges £30 to come to your house plus an extra £20 for each hour that the job takes.
A job takes x hours.
How much does the plumber charge? ...x = 3 ...... £90...

7. What can you say about x if $x + y = 10$ and x is less than y?

...x = 4 ...... y = 6 ...... 4+6 = 10...

9 A piece of rope 60 metres long is cut into two pieces.
One piece is x metres long and the other is y metres long.

Write down two equations.
Each equation should use x, y and 60.

...x + y... $= 60$

$x =$ ...30...

13. If $y = 1 + 4x$ and $x = 3$
then $y =$ ...1 + 4 = 5 x 3 = 15...

If $A = 3r^2$ and r = 4
then A = ...~~3 x 3 x 3~~ 4 + 4 + 4 = 12² = 144...

Sheet PD2.2 – *Commentary on learners' work*

Saira: Comparing fractions and decimals

Saira has a misconception which often goes unrecognised. Most teachers are aware of the tendency to ignore decimal points and treat decimals as if they are whole numbers. For example, many learners obtain answers such as '0.75 > 0.8'.

Here, there is evidence of the reverse tendency, i.e. to say that numbers with more decimal places are smaller in value.

There are two common reasons why learners might believe this. Firstly, they feel that, say, '0.45 is in hundredths' while '0.7 is in tenths'. Thus 0.45 < 0.7 because 'tenths are bigger than hundredths'. Secondly (and this is the reason that is implied here), they believe that 0.45 is analogous or equivalent to $\frac{1}{45}$.

Saira shows in her answers that she understands one meaning of the denominator in a fraction; she sees $\frac{3}{8}$ as involving the cutting of a cake into eight parts. She seems, however, to ignore the value of the numerator when comparing fractions.

Damian: Division and division notation

Damian's answer '5.0' to question 1(a) '4 ÷ 20' suggests that he recognises the concept of division, but that he reads '4 ÷ 20' as 'How many 4s are there in 20?'.

However, this conjecture is not supported by later answers. His answers to question 7 show that he reads '85 ÷ 17' as 'how many 17s go into 85?' and '19 ÷ 76' as 'how many 19s go into 76?'. Thus he appears to reverse his reading of the symbol '÷' to accommodate his feeling that one must always divide the larger number by the smaller.

Notice how question 7 reveals that Damien also has difficulty with other symbols for division. He appears to view '$a\overline{)b}$' and '$a \div b$' as identical in meaning. He may also have a resistance to 'top heavy' fractions.

Questions 1 and 5 also show that Damien has difficulties when estimating the result of division by a decimal less than 1. He produces answers which suggest that he believes that 'division makes numbers smaller'.

Damian appears to ignore the decimal point in questions 1(c) '10 ÷ 0.5' and 1(f) '60 ÷ 0.3'. His rough estimate (17.00) in question 5 '26.12 ÷ 0.286' further suggests that he may think that division of a number by a small quantity reduces that number by a small quantity.

Sheet PD2.2 – *Commentary on learners' work* (continued)

Julia: Perimeter and area

Julia's answers to question 3 show that she is able to distinguish perimeter from area which, in her words, mean 'the outside of a shape' and 'the flat surface of a shape'. She appears to have difficulty explaining the concept of volume, which she describes as the 'whole shape ' including length, surface area and inside the shape. Her drawings indicate that she does relate these concepts to the number of dimensions involved.

Her answers to question 4 indicate that she knows that area is conserved when a shape is cut up and reassembled, but she seems to think that perimeter is also conserved.

Her response to question 5 suggests that she believes that, if the area of a shape is increased, then the perimeter must also increase. She may therefore believe that there is a relationship between the area and perimeter of a shape.

Jasbinder: Algebraic notation

Jasbinder's answers to questions 6, 7 and 9 show that she does not recognise that letters represent variables. In every case, she substitutes particular values for the letter (3, 4 and 30 for x), so there is some realisation that x can take different values in different questions, but she does not allow for this within a single question.

Notice also how she has let $n = 14$ in question 6. This is presumably because n is the 14th letter in the alphabet. This is reminiscent of children's secret codes, where $a = 1$, $b = 2$ and so on.

Her responses reveal a general reluctance to leave operations in answers. She appears to think that, if an operation is present, then something still needs to be done.

Her answer to question 13 shows that she does not recognise the conventions of algebra, e.g. that multiplication precedes addition and that squaring precedes multiplication.

Her use of the 'equals' sign is idiosyncratic. As with many learners, she writes such things as '1 + 4 = 5 × 3 = 15' while evaluating an expression. This tendency is consistent with an interpretation of the '=' symbol as meaning 'makes', ie a signal to evaluate what has gone before. This is the same meaning as the button that has this label on a calculator.

Sheet PD2.3 – *Generalisations made by learners*

What other examples can you add to this list?

Can you think of any mathematical misconceptions you have had at some time?

How did you overcome these?

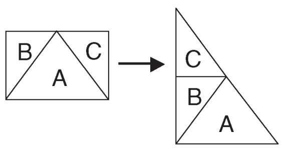
0.567 > 0.85
'The more digits a number has, the larger is its value.'

3 ÷ 6 = 2
'You always divide the larger number by the smaller one.'

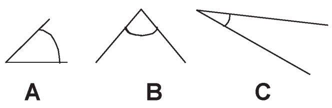
0.4 > 0.62
'The fewer the number of digits after the decimal point, the larger is its value. It's like fractions.'

5.62 × 0.65 > 5.62
'Multiplication always makes numbers bigger.'

1 litre costs £2.60; 4.2 litres cost £2.60 × 4.2; 0.22 litres cost £2.60 ÷ 0.22.
'If you change the numbers in a question, you change the operation you have to do.'



Area of rectangle ≠ area of triangle.
'If you dissect a shape and rearrange the pieces, you change the area.'



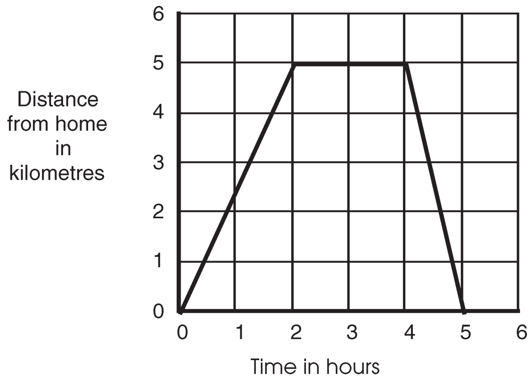
Angle A is greatest. Angle C is greatest.
'The size of an angle is related to the size of the arc or the length of the lines.'

If $x + 4 < 10$, then $x = 5$.
'Letters represent particular numbers'.

3 + 4 = 7 + 2 = 9 + 5 = 14
'"Equals" means "makes"'.

He was going up a steep hill.
'Graphs are just pictures.'

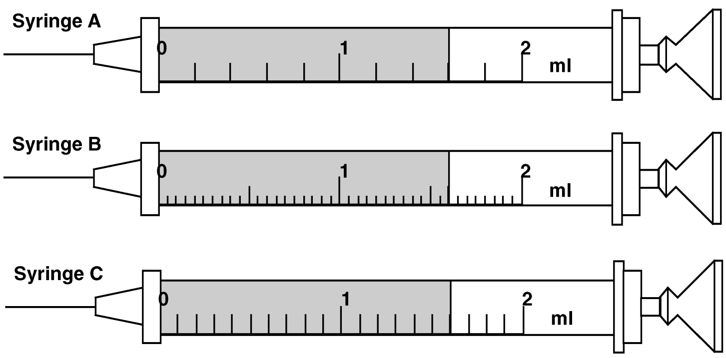
In three rolls of a die, it is harder to get 6,6,6 than 2,4,6.
'Special outcomes are less likely than more representative outcomes.'



Sheet PD2.4 – *Some principles to consider*

The following principles are supported by research evidence.[1] Discuss the implications for your own teaching.

- Teaching approaches that encourage learners to explore misconceptions through discussion result in deeper, longer-term learning than approaches that try to avoid mistakes by explaining, from the start, the 'right way' to see things.
- It is helpful if discussions focus on known difficulties. Rather than posing many questions in one session, it is better to focus on a challenging question and encourage a variety of interpretations to emerge, so that learners can compare and evaluate their ideas.
- Questions can be juxtaposed in ways that create a tension (sometimes called a 'cognitive conflict') that needs to be resolved. Contradictions that arise from conflicting methods or opinions can create an awareness that something needs to be learned. For example, asking learners to say how much medicine is in each of the following syringes may result in answers such as '1.3 ml, 1.12 ml and 1.6 ml'. 'But these quantities are all the same!' This provides the start for a useful discussion on the denary nature of decimal notation.



- Activities should provide opportunities for meaningful feedback. This does not mean providing summative information, such as the number of correct or incorrect answers. More helpful feedback is provided when learners compare results obtained from alternative methods until they realise why they get different answers.
- Sessions should include time for whole group discussion in which new ideas and concepts are allowed to emerge. This requires sensitivity so that learners are encouraged to share tentative ideas in a non-threatening environment.
- Opportunities should be provided for learners to 'consolidate' what has been learned by applying the newly constructed concept.

[1] For a summary of research see Swan M., *Dealing with misconceptions in mathematics*, in Gates P. (ed.), *Issues in mathematics teaching*, pp. 147–165. London, 2001, RoutledgeFalmer.

Sheet PD2.5 – *Looking at one session from the resource*

With a colleague, work through one session from the resource. As you do so, write notes under the following headings.

Number and title of session:

What major mathematical concepts are involved in the activity?

What common mistakes and misconceptions will be revealed by the activity?

How does the activity:

- encourage a variety of viewpoints and interpretations to emerge?
- create tensions or 'conflicts' that need to be resolved?
- provide meaningful feedback?
- provide opportunities for developing new ideas?