MODULE
6

Geometrical reasoning 2

OBJECTIVES

This module is for study by an individual teacher or group of teachers. It:

- considers the connections between loci and constructions;
- discusses activities and resources to develop pupils' visual and geometrical reasoning skills.

CONTENT

The module is in five parts.

1 Introduction

2 Loci defined through distance

3 Generating loci

4 Formal constructions

5 Summary of Modules 5 and 6

RESOURCES

Essential

- Your personal file for inserting resource sheets and making notes as you work through the activities in this module
- The *Framework for teaching mathematics: Years 7, 8 and 9*
- A pencil, ruler, set square and compasses for drawing constructions
- A blank sheet of A4 paper, about 30 cm of string and a 5p coin
- The resource sheets at the end of this module:

  6a 3-D visualisation activities

  6b Moving a coin

  6c Loci problems

  6d Sample teaching unit

  6e Key Stage 3 National Curriculum tests: questions on loci

  6f Summary and further action on Module 6

Desirable

- *Year 9 geometrical reasoning: mini-pack* www.standards.dfes.gov.uk/midbins/keystage3/Y9_geometricreasoning.PDF
- *Teaching and learning geometry 11–19*, a joint report from the Royal Society and the Joint Mathematical Council, available at: www.royalsoc.ac.uk/templates/statements/statementDetails.cfm?StatementID=154
- The QCA *Mathematics glossary for teachers in Key Stages 1 to 4*, available at www.qca.org.uk/ages3-14/downloads/glossary.pdf

STUDY TIME

Allow approximately 90 minutes.

Part 1 Introduction

1 Module 6 looks at ways of establishing a deeper understanding of the links between loci and constructions through visualisation and practical activity. The aim is to build on the deductive reasoning skills discussed in Module 5.

2 The two modules on geometrical reasoning concentrate mainly on examples in two dimensions. Pupils' reasoning skills also need to be developed in three-dimensional contexts.

We will therefore start with a 3-D visualisation activity. Try the two activities on **Resource 6a, 3-D visualisation activities**.

3 In the two activities on Resource 6a, the new shape formed by joining the centre of each face to the centre of adjacent faces is called the *dual* of the first shape. Compare your answers with those below.

A square-based pyramid has 5 faces, 5 vertices and 8 edges. The 5 faces will generate 5 vertices for the new shape. If we assume that the original pyramid as standing on its square base, then the dual is a smaller, different square-based pyramid, hanging upside down inside the original shape.

A cube has 6 faces, 8 vertices and 12 edges. The dual of the cube has 6 vertices and 8 faces. It is an octahedron but could also be described as two square-based pyramids sharing a common square base and pointing in opposite directions.

You probably noticed that the number of vertices and edges in the dual corresponds to the number of faces and edges in the original shape. In your follow-on questions, did you include things like: 'What is the dual of a tetrahedron?' and 'When is the dual a similar shape to the original?'

4 Now study the 3-D examples in the supplement of examples, Framework section 4, pages 198–201. Make a note in your personal file of any examples that would be useful to explore with the classes that you teach. Add to your notes two or three follow-on questions to ask pupils when they have completed the activity.

Part 2 Loci defined through distance

1 We will now look at the idea of a locus. The QCA glossary defines a **locus** as:

- the set of points that satisfy given conditions.

For example, in three dimensions, the locus of all points that are a given distance from a fixed point is a sphere.

2 Try the three problems on **Resource 6b, Moving a coin**. For these you will need a blank sheet of A4 paper, about 30 cm of string, a 5p coin, a pencil, a ruler, a set square and compasses.

3 Think about the problems you have just done. Compare your solutions with those below.

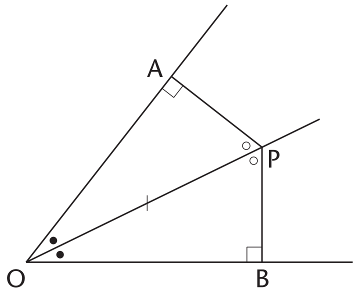
Locus 1: The locus is a line parallel to the long edge of the paper at a perpendicular distance of 5 cm from it.

Locus 2: The locus is the bisector of the right angle at the corner of the paper.

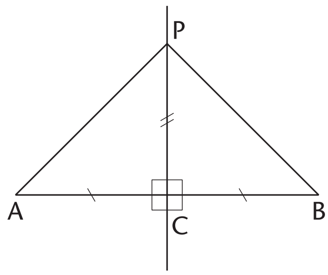
Locus 3: The locus is the perpendicular bisector of the line segment joining the two fixed points. One way of thinking about the moving point is to see it as the apex of an isosceles triangle.

How would you justify your solutions to these problems? In **locus 1**, the justification lies in the definition of parallel lines – they are always the same distance apart.

In **locus 2**, we have to justify that a point P on the bisector of angle AOB is always the same distance from each of the two arms of the angle. Informally, the angle bisector OP is a line of symmetry, so the two triangles AOP and BOP are identical. More formally, triangles AOP and BOP are congruent (two angles and the included side are the same). It follows that AP = PB.



In **locus 3**, we have to justify that a point P on the perpendicular bisector of the line joining two points A and B is always the same distance from each of the two points. Here again, we can use informal arguments based on symmetry, or we can prove that triangle PAC and PBC are congruent (two sides and the included angle). It follows that PA = PB.



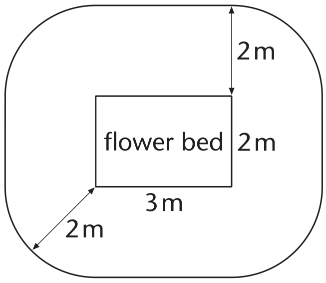
Part 3 Generating loci

1 It is often easier to think about loci by working on them practically. Exploration of problems in the classroom can sometimes be supported by asking a pupil to make moves according to the instructions given by another pupil. Other pupils can check using a ball of string whether the conditions of the original problem are being met. For example, locus 3 above could be generated by asking a pupil to move so that she remains an equal distance from each of two pupils who are standing still. Even when exact calculations are not made, it is possible to deduce properties that link with familiar shapes.

2 Try the practical investigations on **Resource 6c, Loci problems**.

When you have finished, compare your solutions to the problems on Resource 6c with the notes below.

Locus 4: It is easy to assume that the locus of points 2 metres from the flower bed is a rectangle. Closer examination of what happens at the corners is needed!



Locus 5: This generates an ellipse.

The axis of symmetry parallel to AB or 'major axis' is 10 cm long, the same as the sum of the distances of point P from A and B; the axis of symmetry perpendicular to AB is 8 cm long, found using Pythagoras' theorem.

As the pins move towards each other, the shape becomes closer to a circle. As the pins move further apart (the maximum possible distance between them is 10 cm), the shape becomes closer to the straight line AB.

Part 4 Formal constructions

1 Informal work on loci is a useful precursor to work on straight edge and compass constructions. It can provide motivation for the more formal work and encourage pupils to look for reasons why a particular method works or sometimes to discover a method for themselves.

The formal constructions are unlike the constructions that pupils have done using a ruler and protractor in Key Stage 2 and Year 7. The constructions done earlier all involve measurements in one way or another and so have a degree of imprecision. In contrast, straight edge and compass constructions are, at least in principle, exact.

What formal constructions do you remember doing when you were at school? Here are some that you might remember.

- Construct the mid-point of a given line segment.
- Construct the perpendicular bisector of a given line segment.
- Construct the bisector of a given angle.
- Construct the perpendicular to a given line segment from a given point not on the line.
- Construct the perpendicular to a given line segment at a given point on the line.

You will need a **pencil**, **straight edge** and **compasses**. In your personal file, draw as many as you can of the constructions above.

As you work, think about what you are doing, and *why* – for example, by considering as you draw why any arcs and points are marked as they are.

When you have finished, check what you have done by looking at the supplement of examples, Framework section 4, page 221, focusing on the examples for Year 8.

2 Think about how each of the constructions on Framework page 221 links to the properties of a rhombus. Here are some possibilities.

- Diagram 1: the diagonals of a rhombus bisect at right angles.
- Diagram 2: the diagonals of a rhombus bisect the interior angles.
- Diagram 3: the diagonals of a rhombus intersect at right angles.
- Diagram 4: the diagonals of a rhombus intersect at right angles.

It is important to help pupils to make connections between different aspects of mathematics. Here, for example, there is an opportunity for pupils to appreciate how familiar properties of the rhombus can be applied to solving problems on construction.

3 Look at the references to constructions in the teaching programmes for Years 7, 8 and 9, Framework section 3, pages 7, 9, 11. Look also at the linked examples in the supplement of examples, section 4, page 220–223.

As you study these references and examples, consider these points.

- Where do objectives involving constructions appear?
- Are there any key objectives? If so, in which year?
- What examples are illustrated in the supplement? Would any of these be useful to incorporate into mathematics lessons for the classes that you teach? If so, make a note of them in your personal file.

4 Look at the sample unit 'Construction and loci (8S1)' in **Resource 6d, Sample teaching unit.** The department using this unit decided to teach loci before constructions to allow pupils to learn from the practical experience of generating sets of points and to make connections between this and their work on constructions.

Read the sample unit and match it to the objectives for 'Construction and loci' in the Year 8 teaching programme (Framework section 3, page 9).

Unit plans don't have to be presented like the example in Resource 6d. There are many different ways of recording them effectively, although it is likely that they all illustrate the same important features. Try to identify what these important features are, then check how they appear in the scheme of work for your own school.

5 Look at **Resource 6e, Key Stage 3 National Curriculum tests: questions on loci**. The questions are all at level 7.

What skills and knowledge would pupils need in order to answer the questions successfully?

For each test question, think about the kinds of informal or formal arguments that you would expect pupils to give if asked to justify their reasoning.

Part 5 Summary of Modules 5 and 6

1 Pupils can be aware of and use geometrical facts or properties that they have discovered intuitively from practical work before they can prove them analytically. The aim in Key Stage 3 is for pupils to use and develop their knowledge of shapes and space to support geometrical reasoning. Teach them to understand and use short chains of deductive reasoning and results about alternate and corresponding angles to

reach a proof. Later, pupils should be able to explain why the angle sum of any quadrilateral is 360°, and to deduce formulae for the area of a parallelogram and of a triangle from the formula for the area of a rectangle. These chains of reasoning are essential steps towards the proofs that are introduced in Key Stage 4.

Geometry cannot be learned successfully solely as a series of logical results. Pupils also need opportunities to use instruments accurately, draw shapes and appreciate how they can link together or be dissected. It is vital to distinguish between the imprecision of constructions that involve protractors and rulers, and the 'exactness in principle' of standard constructions that use only compasses and a straight edge. Geometrical reasoning can show pupils why construction methods work – for example, the method to construct a perpendicular bisector of a line segment.

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2 Look back over the notes you have made during this module. Have you identified what you may need to consider and adopt in your planning and teaching of geometry?

Use **Resource 6f, Summary and further action on Module 6**, to list key points you have learned, points to follow up in further study, modifications you will make to your planning or teaching, and points to discuss with your head of department.

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3 If you are interested in reading more about the teaching of geometry in secondary schools, read *Teaching and learning geometry 11–19*, a joint report from the Royal Society and the Joint Mathematical Council. This report reiterates the centrality of geometry to the mathematics curriculum and how important it is that this branch of the subject should not be neglected. You can download the report from www.royalsoc.ac.uk/templates/statements/statementDetails.cfm?StatementID=154.

If you have not already done so, you could also download and look at the Key Stage 3 Strategy's *Year 9 geometrical reasoning: mini-pack* from www.standards.dfes.gov.uk/midbins/keystage3/Y9_geometricreasoning.PDF.

Resource 6a 3-D visualisation activities

The first part of each of these activities should be carried out without any drawing.

1 SKELETON PYRAMID

Imagine a wire-framed skeleton of a square-based pyramid.

How many faces does the pyramid have?

How many edges are there? How many vertices?

Imagine locating the centre of each face.

Imagine each centre joined to the centres of its adjacent faces.

The lines joining these centres form the skeleton of another 3-D shape.

Write the name of this new shape in the box below.

The new shape is a …………………………………………………………………………..

2 SKELETON CUBE

Imagine a wire-framed skeleton of a cube.

How many faces does the cube have?

How many edges are there? How many vertices?

Imagine locating the centre of each face.

Imagine each centre joined to the centres of its adjacent faces.

The lines joining these centres form the skeleton of another 3-D shape.

Write the name of this new shape in the box below.

The new shape is a …………………………………………………………………………..

[continued on the next page]

In the box below, jot down some questions that you could ask pupils that would follow on from this visualisation activity.

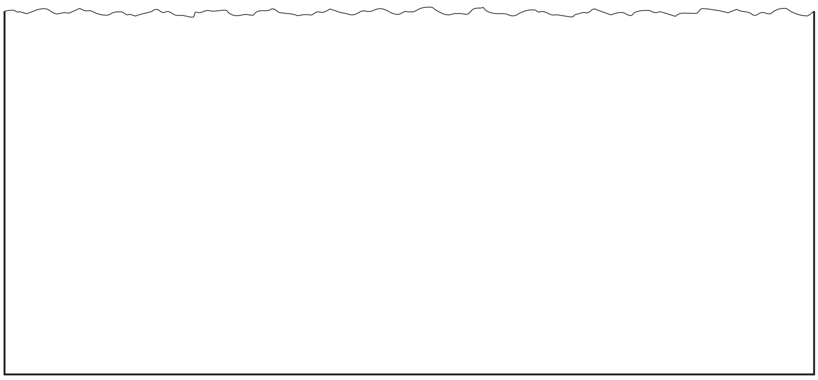
Resource 6b Moving a coin

For these activities you will need a blank sheet of A4 paper, about 30 cm of string, a 5p coin, a pencil, a ruler, a set square and compasses.

LOCUS 1

Place the 5p coin so that its centre is about 5 cm from a long edge of the paper.
Push the coin so that it is always the same distance from the long edge of the paper.
Use the string to check that the distance from the edge stays constant.

Make a freehand sketch of the path of the centre of the coin on the rectangle below.



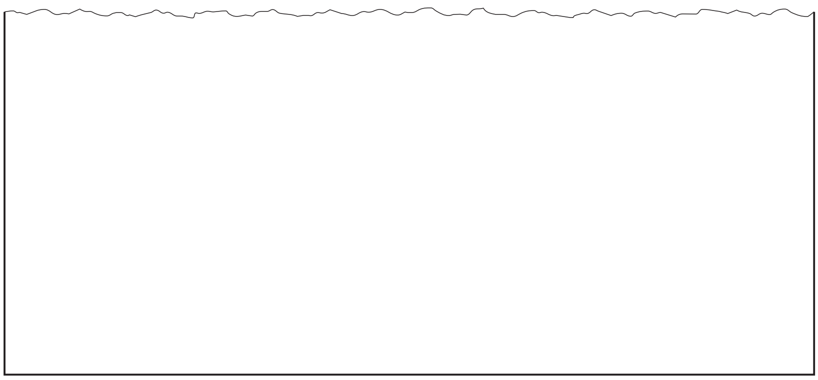
Write a description of the path of the coin.

The coin moves along a line that is:

LOCUS 2

Place the 5p coin so that its centre is an equal distance from a long edge and a short edge of the paper.
Push the coin so that its centre is always the same distance from the long and short edges.
Use the string to check that the distances remain the same.

Make a freehand sketch of the path of the coin on the rectangle below.



Write a description of the path of the coin.

The coin moves along a line that is:

LOCUS 3

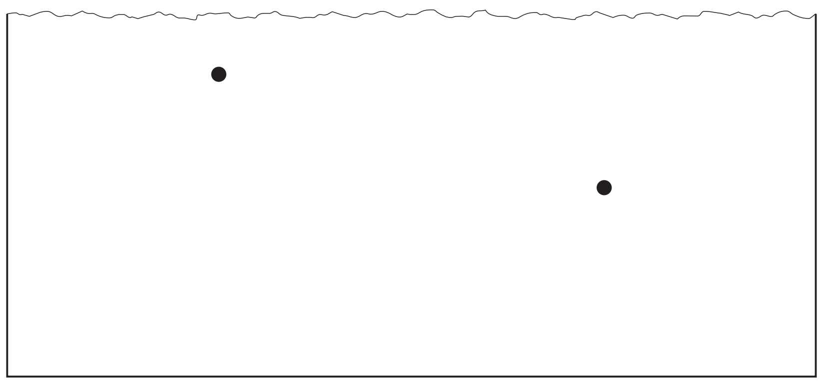
Mark two points on the A4 paper, about 8 cm apart.

Place the 5p coin so that it is an equal distance from each of the two points.

Push the coin so that it is always the same distance from the two points.

Use the string to check that the measurements are correct.

Make a freehand sketch of the path of the coin on the rectangle below.



Write a description of the path of the coin.

The coin moves along a line that is:

LOCUS 4

A dog is trained to walk around a rectangular flower bed in a garden so that it always remains 2 metres from the edge of the flower bed.
The flower bed is 2 metres wide by 3 metres long.

Draw the locus of points along which the dog walks. Think carefully about what happens at the corners.

Write a description of the locus using correct mathematical terms.

LOCUS 5

Two fixed points A and B are 6 cm apart.

Draw the locus of points P so that the total distance A to P plus P to B is always 10 cm. A ruler, or even a piece of string and pins at A and B, might help.

What will happen to this locus as A and B move closer together?

What will happen to this locus as A and B move further apart?

Construction and loci (8S1; 6 hours)

This Year 8 unit on Shape, space and measures follows on from the Year 7 unit 7S5, in which pupils constructed triangles using a ruler and protractor (SAS and ASA). In this unit, pupils are introduced to the notion of a locus through practical tasks (using people and/or counters on the OHP). This is developed into constructions, making links clear. Pupils should be encouraged to use their reasoning skills to draw out the properties of shapes (e.g. diagonals of a rhombus) and to do the four standard constructions.

Teaching objectives for the oral and mental starters

- Estimate and order acute, obtuse and reflex angles.
- Visualise, describe and sketch 2-D shapes and describe their properties, including symmetries.
- Visualise and describe the effects of reflections and translations on 2-D shapes.
- Visualise simple loci.
- Multiply and divide integers and decimals by 10, 100 and 1000 and explain the effect (converting between metric units in preparation for unit 8S2).
- Select some mental and oral number skills from previous work to reinforce learning.

Teaching objectives for the main activities

Simplification (Y7 objectives)	**Core** (mainly Y8 objectives)	**Extension** (Y9 objectives)
• Use a ruler and protractor to measure and draw lines to the nearest millimetre and angles, including reflex angles, to the nearest degree.	• Use straight edge and compasses to construct: – the mid-point and perpendicular bisector of a line segment; – the bisector of an angle; – the perpendicular from a point to a line; – the perpendicular from a point on a line.	
• Use a ruler and protractor to construct a triangle given two sides and the included angle (SAS) or two angles and the included side (ASA); explore these constructions using ICT.	• Construct a triangle given three sides (SSS). • Use ICT to explore these constructions. • Find simple loci, both by reasoning and by using ICT, to produce shapes and paths, e.g. an equilateral triangle. • Use correctly the vocabulary, notation and labelling conventions for lines, angles and shapes. • Present and interpret solutions, justifying inferences and explaining reasoning, using step-by-step deduction; give solutions to an appropriate degree of accuracy in the context of the problem.	• Use straight edge and compasses to construct a triangle, given right angle, hypotenuse and side (RHS). • Use ICT to explore constructions of triangles and other 2-D shapes. • Explain how to find, calculate and use the sums of the interior and exterior angles of regular polygons.

Activities and key teaching points

Oral and mental starters	Notes
Use questions that require pupils to order a range of angles (acute, obtuse, reflex).	
Use questions that require pupils to visualise and describe shapes and their properties, e.g. reflections, translations, rotations.	Ensure that some questions include 3-D shapes.
Use questions that require pupils to visualise and describe paths to generate loci.	See Framework examples page 225.
Practise conversion of metric units: mm to cm, cm to m, m to km, and vice versa.	
Practise mental addition and subtraction of decimals (two-digit numbers).	
Main activities	
Simple loci	
Use practical tasks to develop pupils' appreciation of locus. Emphasise the use of simple loci to produce shapes and paths.	The word 'locus' comes from the Latin word for 'place'. Locus may initially be thought of as the path traced out by a point as it moves under given conditions. It should then be defined as the set of all points that satisfy these given conditions.
Activity: Understanding locus as the path traced out by a point.	Framework examples pages 224–227.
Activity: Place counters on a table according to a given rule and determine the locus generated.	Framework examples pages 224–227.
Both these activities can be demonstrated effectively using counters on an OHP, using pupils as points or counters, or using dynamic geometry software.	Using a computer and large screen, ICT can be used to generate shapes and paths. For more able pupils, extend this work to include Year 9 examples.
Constructions	
Emphasise the use of compasses to construct all points at a fixed distance from a point. Use examples such as:	Construction of equilateral triangles may be appropriate here.
• Given a line AB, ask pupils to show the points that are 3 cm from A and 4 cm from B. Use this as an introduction to drawing triangles given three sides (SSS) using compasses.	Link to locus examples. See Framework examples page 221.

• Ask pupils to construct a rhombus using a straight edge and compasses, given the length of a diagonal and the length of a side. Provide an opportunity for pupils to review the properties of the rhombus.	In Year 7 pupils constructed a rhombus given the length of one side and one of the angles. Pupils should use their knowledge of the properties of the rhombus to explain why the construction works.
Activity: Use straight edge and compasses for constructions. Teach pupils to construct: • the perpendicular from a point to a line; • the perpendicular from a point on a line; • the midpoint and perpendicular bisector of a line segment; • the bisector of an angle. Give pupils a range of examples to use these constructions, e.g. from the textbook, worksheet and OHTs available in the department resource file.	Framework examples pages 220–223.
Constructions and loci	
Use ICT software (dynamic geometry and Logo) to generate shapes and paths.	Framework examples pages 224–227.
Activity: Use ICT to generate shapes and paths. Ask pupils to construct rectilinear shapes, regular polygons and equi-angular spirals. Ask pupils to explore the construction of regular polygons using Logo. Ask pupils to produce nets for regular tetrahedron and octahedron.	Framework examples pages 224–227. Year 9 objectives include the calculation of internal and external angles of regular polygons. More able pupils should generalise results while others continue to experiment.

Resources

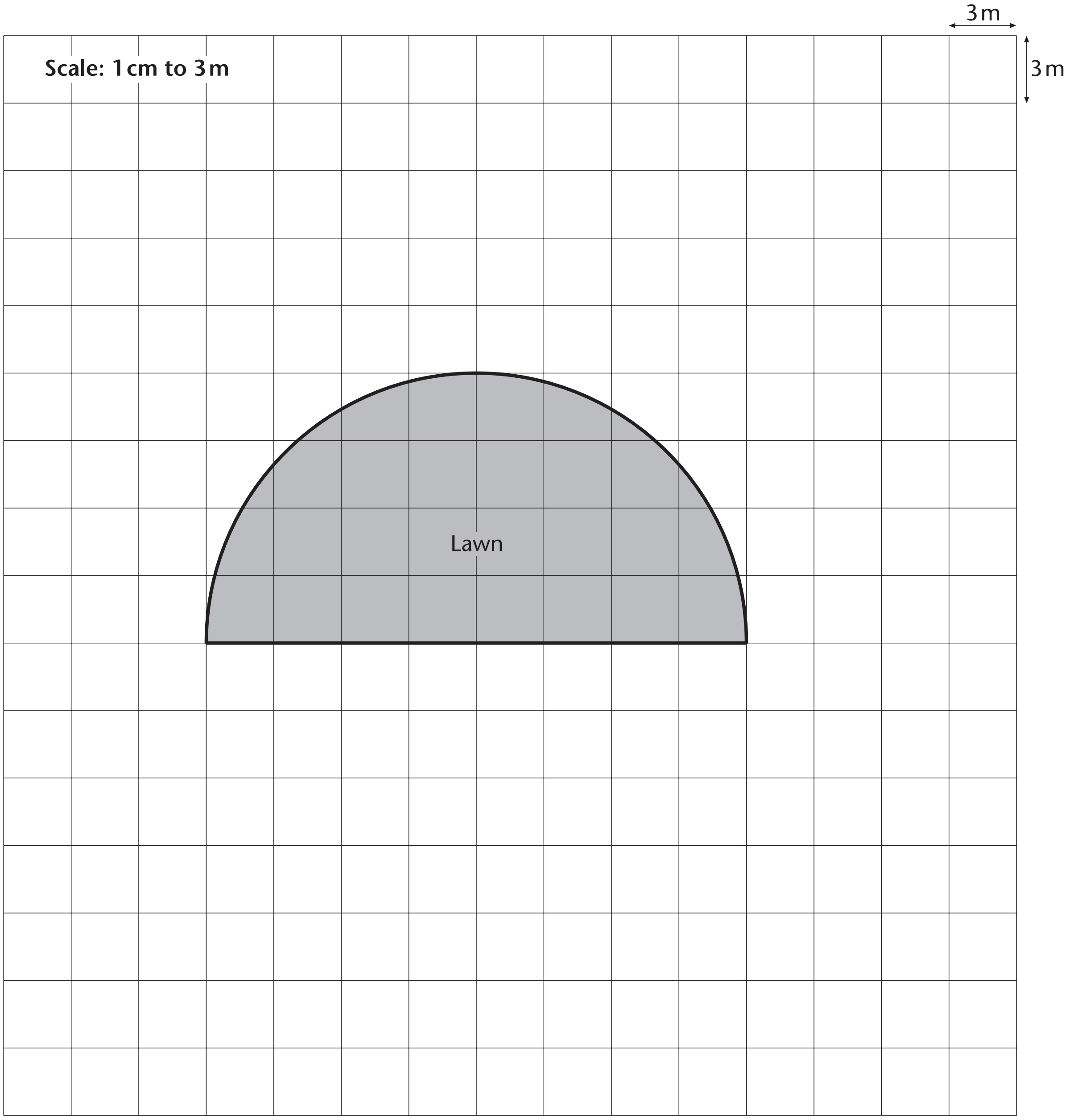
Counters, OHP, rulers, compasses, plain paper, ICT software (dynamic geometry, Logo)

Key vocabulary

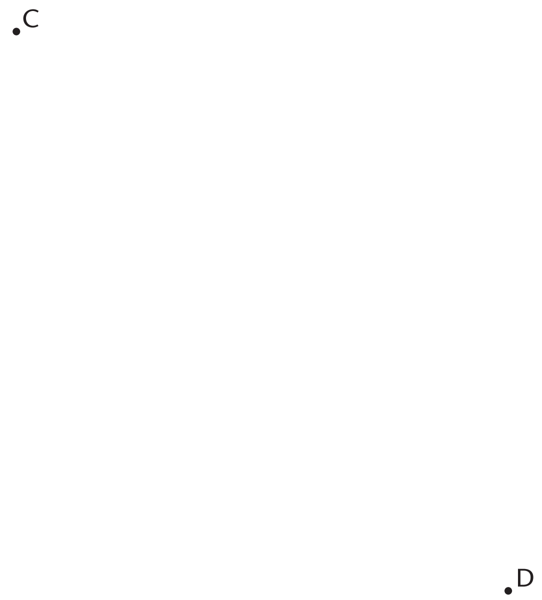
Bisect, bisector, perpendicular bisector, mid-point, line segment, equidistant, compasses, locus, loci

Resource 6e Key Stage 3 National Curriculum tests: questions on loci

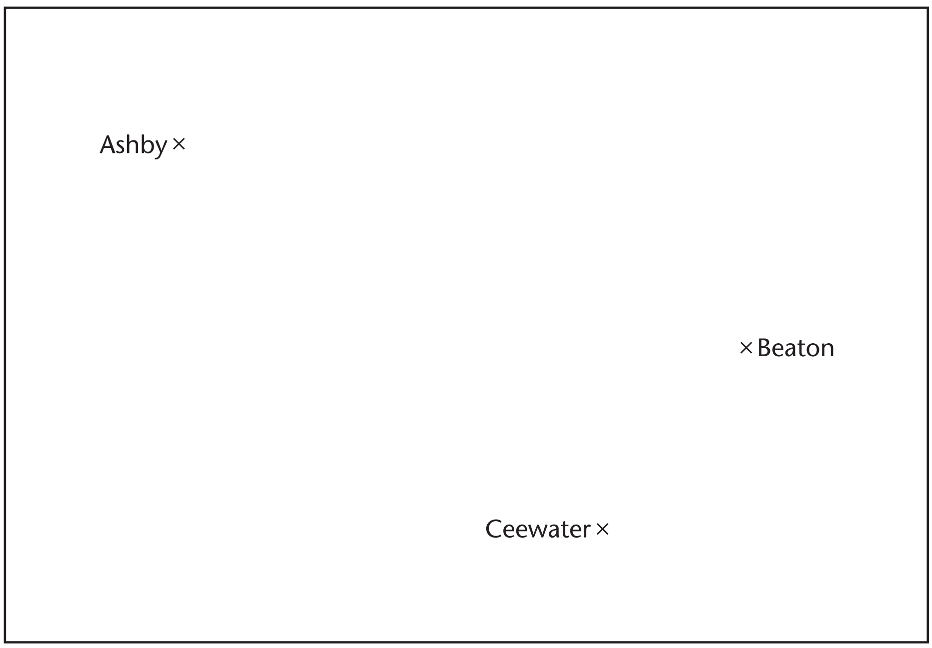
1 In the scale drawing, the shaded area represents a lawn.
There is a wire fence all around the lawn.
The shortest distance from the fence to the edge of the lawn is always 6m.
On the diagram, draw accurately the position of the fence.



2 Look at points C and D below.
Use a straight edge and compasses to draw the locus of all points that are the same distance from C as from D.
Leave in your construction lines.



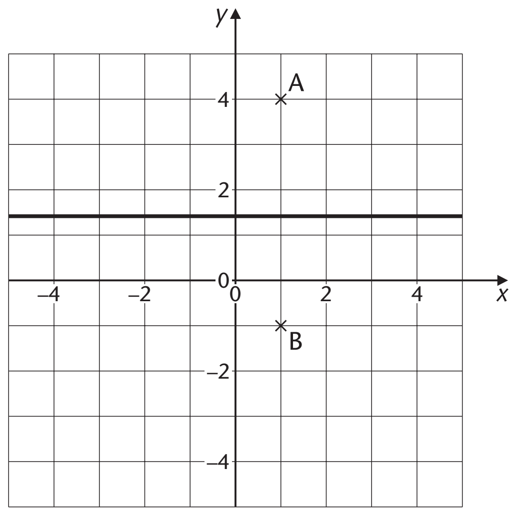
3 The plan shows the position of three towns, each marked with a ×.
The scale of the plan is 1 cm to 10 km.



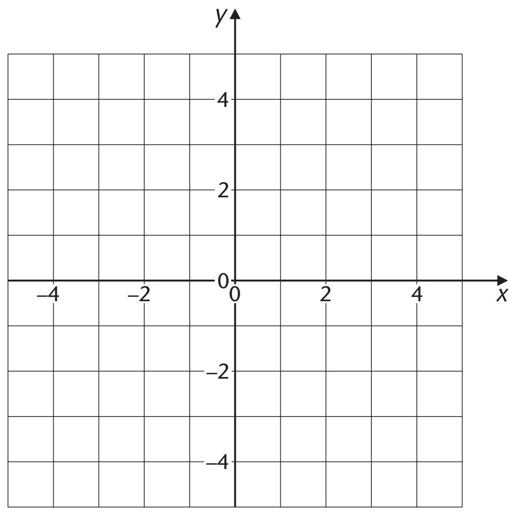
The towns need a new radio mast. The new radio mast must be:
nearer to Ashby than Ceewater,
and less than 45 km from Beaton.

Show on the plan the region where the new radio mast can be placed.
Leave in your construction lines.

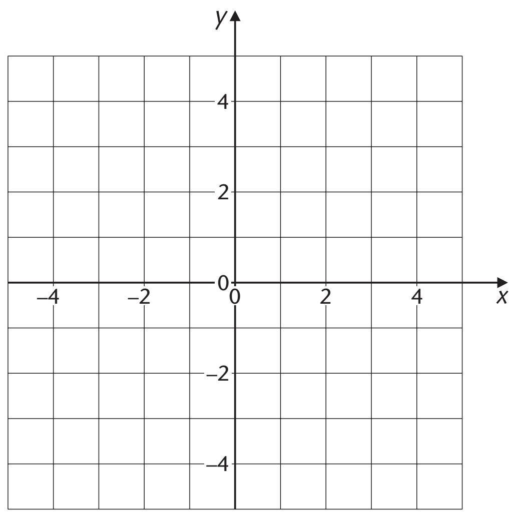
4 The diagram shows the locus of all points that are the same distance from A as from B. The locus is one straight line.



The locus of all points that are the same distance from (2, 2) and (–4, 2) is also one straight line. Draw this straight line.



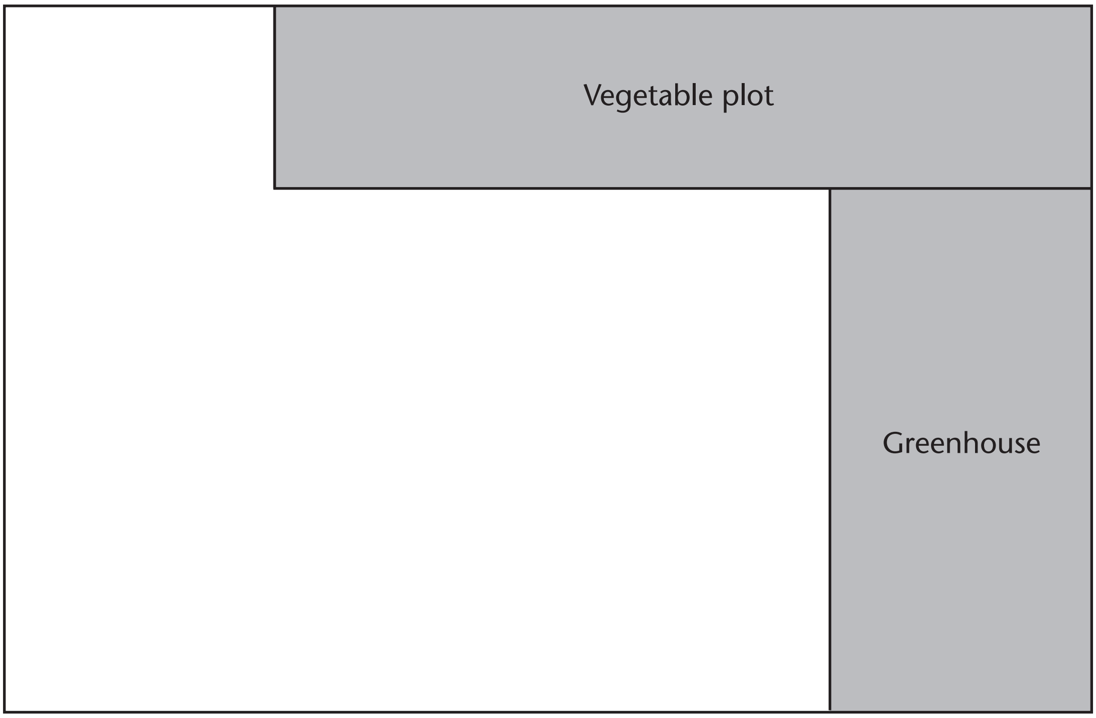
The locus of all points that are the same distance from the *x*-axis as they are from the *y*-axis is two straight lines. Draw both straight lines.



5 A gardener wants to plant a tree.
She wants it to be more than 8 m away from the vegetable plot.
She wants it to be more than 18 m away from the greenhouse.

The plan below shows part of the garden.
The scale is 1 cm to 4 m.

Show accurately on the plan the region of the garden where she can plant the tree.
Label this region R.



Resource 6f Summary and further action on Module 6

Look back over the notes you have made during this module. Identify the most important things to consider and modify in your planning and teaching of geometry.

List two or three key points that you have learned.

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List two or three points to follow up in further study.

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List two or three modifications that you will make to your planning or teaching of geometry.

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List the most important points that you want to discuss with your head of department, or any further actions you will take as a result of completing this module.

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