MODULE 7

Ratio and proportion 1

OBJECTIVES

This module is for study by an individual teacher or group of teachers. It:

- discusses ratio and proportion as key mathematical ideas, with applications across many aspects of Key Stage 3 mathematics and in other subjects;
- analyses a lesson on ratio and proportion.

CONTENT

The module is in five parts.

1 Introduction

2 Fractions, ratio and proportion

3 Looking at a lesson on ratio and proportion

4 Proportion as a functional relationship

5 Summary

RESOURCES

Essential

- Your personal file for inserting resource sheets and making notes as you work through the activities in this module
- The *Framework for teaching mathematics: Years 7, 8 and 9*
- Video sequence 4, a Year 7 lesson on ratio and proportion, from the DVD accompanying this module, and a DVD player
- The resource sheets at the end of this module:
  - 7a Extract adapted from section 1 of the Framework
  - 7b Introducing ratio and proportion
  - 7c Walt's lesson plan
  - 7d Observing Walt's lesson
  - 7e Proportions and graphs
  - 7f Proportion as a functional relationship
  - 7g Summary and further action on Module 7

Desirable

- *What is a fraction?*
  www.standards.dfes.gov.uk/midbins/keystage3/ma_fraction000602.PDF
- *Year 7 fractions and ratio: mini-pack*
  available on the Key Stage 3 website from August 2004
- *Year 8 multiplicative relationships: mini-pack*
  www.standards.dfes.gov.uk/midbins/keystage3/multip_minipack_pupilres.PDF
- *Year 9 proportional reasoning: mini-pack*
  www.standards.dfes.gov.uk/midbins/keystage3/Y9_propreasoning_mini.PDF

STUDY TIME

Allow approximately 75 minutes, plus reading time for *What is a fraction?*

Part 1 Introduction

1 Before you study this module, aim to download and read ***What is a fraction?*** www.standards.dfes.gov.uk/midbins/keystage3/ma_fraction000602.PDF.

The topic of ratio and proportion is a key aspect of the number curriculum in Key Stage 3. It addresses ideas that permeate all branches of mathematics and has many applications in other subjects.

In your personal file, jot down some applications of ratio and proportion which pupils are likely to meet at Key Stage 3, in mathematics and in other subjects.

Read **Resource 7a, Extract adapted from section 1 of the Framework**.

2 The topic of ratio and proportion is often regarded as one that gives rise to some difficulties in teaching and learning. Modules 7 and 8 consider:

- how the underlying ideas can be clarified;
- how links between different aspects can be drawn out.

Part 2 Fractions, ratio and proportion

1 Pupils first meet fractions as parts of whole numbers – particularly in relation to the operation of sharing or dividing into equal parts. In Years 5 and 6, pupils working at levels 4 and 5 meet the ratio aspect of fractions. This aspect features more prominently in Key Stage 3.

Read and do the activities in **Resource 7b, Introducing ratio and proportion**.

2 The proportional sets that can be generated from the set of integers 1 to 10 are:

$\frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{5}{5} = \frac{6}{6} = \frac{7}{7} = \frac{8}{8} = \frac{9}{9} = \frac{10}{10}$

$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10}$ and $\frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{8}{4} = \frac{10}{5}$

$\frac{1}{3} = \frac{2}{6} = \frac{3}{9}$ and $\frac{3}{1} = \frac{6}{2} = \frac{9}{3}$

$\frac{2}{3} = \frac{4}{6} = \frac{6}{9}$ and $\frac{3}{2} = \frac{6}{4} = \frac{9}{6}$

$\frac{1}{4} = \frac{2}{8}$ and $\frac{4}{1} = \frac{8}{2}$

$\frac{3}{4} = \frac{6}{8}$ and $\frac{4}{3} = \frac{8}{6}$

$\frac{1}{5} = \frac{2}{10}$ and $\frac{5}{1} = \frac{10}{2}$

$\frac{2}{5} = \frac{4}{10}$ and $\frac{5}{2} = \frac{10}{4}$

$\frac{3}{5} = \frac{6}{10}$ and $\frac{5}{3} = \frac{10}{6}$

$\frac{4}{5} = \frac{8}{10}$ and $\frac{5}{4} = \frac{10}{8}$

3 In summary:

- **Ratio** is a way of comparing two or more quantities measured in the same units – the quantities may be separate entities or they may be different parts of a whole.
- The definition of **proportion** given here, 'an equality of ratios', is an essential mathematical term. Often we use the same word to refer to a fractional part of a whole (for example, the proportion of pupils in a class who are absent). This can sometimes be a source of confusion.

- If two ordered sets of numbers {a, b, c, …} and {q, r, s, …} are such that $a:q = b:r = c:s = \ldots$ (or $\frac{a}{q} = \frac{b}{r} = \frac{c}{s} = \ldots$), then the two ordered sets are said to be in **direct proportion**.

Part 3 Looking at a lesson on ratio and proportion

1 In this part of the module, you will watch a lesson on ratio and proportion. Walt teaches the third of a series of lessons on this topic to a top set in Year 7. Study **Resource 7c, Walt's lesson plan**, before you watch the lesson.

2 Get ready to watch **Video sequence 4, a Year 7 lesson on ratio and proportion**. The teacher is Walt. The lesson has three distinct parts. Watch each part of the lesson separately: first the starter, then the main part of the lesson, and then the final plenary. After each part, consider and make notes on the questions on **Resource 7d, Observing Walt's lesson**.

The whole video sequence lasts about 13 minutes: 4 minutes for the starter, 4 minutes for the main activity and 5 minutes for the plenary.

When you have finished watching, spend a few minutes completing the notes you have made on Resource 7d.

3 Did you notice these features of Walt's lesson?

Starter

- Walt makes effective use of the OHP to review work from the previous lesson.
- Key vocabulary is displayed and Walt uses the terms associated with ratio and proportion very precisely.
- The sequence with the counting stick practises the skill of generating equivalent ratios, a skill which will be used in the main part of the lesson.

Main activity

- Pupils find their own ways of adapting the approach of finding equivalent ratios to particular examples.
- Walt provides a helpful model for recording the steps of the calculation, a model the pupils readily take on board.

Plenary

- Walt relates the same mathematical ideas to a completely different context.
- The context is introduced in an imaginative way and hints at aspects of work which pupils will meet in the future, for example, scale and enlargement.

Part 4 Proportion as a functional relationship

1 Because proportions are a particular type of functional relationship, they will occur in algebra as part of work on functions and graphs. Read **Resource 7e, Proportions and graphs**.

2 Think about and make notes on your answers to the questions on **Resource 7f, Proportion as a functional relationship**.

3 Study the yearly teaching programmes and supplements of examples on ratio and proportion in the Framework, sections 3 and 4. Relevant references for the supplement of examples are in the yearly teaching programmes.

Focus on the progression across Years 7, 8 and 9. Look for ways in which explicit links between different objectives might be made.

As you study, make notes in your personal file on relevant points in your school's scheme of work when you can strengthen pupils' awareness of the links. Note also any points that you want to discuss later with your head of department.

Part 5 Summary

1
- A proportion is an equality of ratios.
- Equivalent fractions (or ratios) are formed from proportional sets of numbers, in which there is a constant multiplier relating numerators to denominators.
- Where two or more sets of numbers represented by the variables (x, y) are such that $\frac{y}{x} = m$, or $y = mx$, where m is a constant, the numbers are in direct proportion, and m is called the constant of proportionality.
- Where x and y are variable numbers, m describes the rate of change of y with respect to x, that is, y changes by m for every 1 of x. The phrases 'for every', 'in every' and 'to every', or simply 'per', are all used when describing a constant rate –for example, miles per hour.
- m is sometimes thought of as a scale factor.
- The graph of $y = mx$ consists of points lying in a straight line through the origin. Graphically, the rate of change is represented by 'rise ÷ step', and m is the gradient of the graph.

2 Look back over the notes you have made during this module. Have you identified what you may need to consider and adopt in your planning and teaching of ratio, proportion and proportional reasoning?

Use **Resource 7g, Summary and further action on Module 7**, to list key points you have learned, points to follow up in further study, modifications you will make to your planning or teaching, and any points to discuss with your head of department.

3 You may find it useful to download and study the Key Stage 3 mini-packs that support the teaching of ratio and proportion:

- *Year 7 fractions and ratio: mini-pack*
  available on the Key Stage 3 website from August 2004
- *Year 8 multiplicative relationships: mini-pack*
  www.standards.dfes.gov.uk/midbins/keystage3/multip_minipack_pupilres.PDF
- *Year 9 proportional reasoning: mini-pack*
  www.standards.dfes.gov.uk/midbins/keystage3/Y9_propreasoning_mini.PDF

Resource 7a Extract adapted from section 1 of the Framework

This short article is based on pages 12 and 13 of section 1 of the Framework.

Proportional reasoning in Key Stage 3

Throughout Key Stage 3 pupils will extend their understanding of the number system to positive and negative numbers and, in particular, to fractions and their representations as terminating or recurring decimals.

Fractions, decimals, percentages, ratio and proportion are different ways of expressing related ideas and relationships. The connections start to be established in Key Stage 2, particularly the equivalence between fractions, decimals and percentages. The ideas of ratio and proportion, and the relationship between them, should be a strong feature of work in Key Stage 3. By the end of the key stage, pupils should be able to solve problems involving fractions, decimals, percentages, ratio and proportion, and their interconnections.

After calculation, the application of proportional reasoning is the most important aspect of elementary number. Proportionality underlies key aspects of number, algebra, shape, space and measures, and handling data. It is also central in applications of mathematics in subjects such as science, technology, geography and art. The study of proportion begins in Key Stage 2 but it is in Key Stage 3 where secure foundations need to be established.

Problems involving proportion are often solved by informal methods, particularly when the numbers involved are easy to deal with mentally. But it is important to teach methods that can be applied generally. For example, the unitary method is useful for solving problems involving proportion, and multiplicative methods involving fractions or decimals are useful for solving percentage problems.

Teaching proportional reasoning

When you are teaching proportional reasoning:

- emphasise the language and notation of ratio and proportion, and the links to fractions, decimals and percentages;
- teach pupils specific methods for solving proportion problems so that they do not remain dependent on informal approaches;
- help pupils to understand what they are calculating: for example, a distance divided by a time gives a speed – an example of a rate; but a distance divided by another distance gives a scale factor or multiplier – a dimensionless number;
- make explicit links between ideas of proportionality in number, algebra, shape, space and measures, and handling data.

Links with other mathematical topics

In algebra, direct proportion is viewed as a linear relationship of the form $y = mx$. The graphical representation of this equation helps pupils to visualise ideas such as rate of change and gradient. The algebraic representation of a proportion (e.g. $a : b = c : d$ or $a/b = c/d$) underpins a general method for solving problems.

In shape, space and measures, proportionality arises when enlargement by different scale factors is considered. Scaling has a wide range of applications, for example, in maps, plans and scale drawings. Similar figures have sides or dimensions that are in proportion. Recognition of the similarity of all circles leads to an understanding that the circumference is directly proportional to the diameter, while awareness of the similarity of triangles with the same angles leads to an understanding of trigonometry.

In statistics, proportions are often calculated when data are interpreted and inferences drawn. Proportions are also used when probabilities are estimated or calculated based on outcomes that, in theory, are equally likely.

Applications of ratio and proportion

Some of the many possible applications of ratio and proportion that pupils are likely to meet or could be introduced to in Key Stage 3, either in mathematics or in other subjects, are as follows.

- The cost of vegetables in the market, at a fixed price per kilogram, is proportional to the weight of vegetables purchased.
- A percentage increase or decrease is a proportion of the original amount or quantity.
- The distance between two points on a map drawn to scale is proportional to the actual distance on the ground.
- Athletes often try to improve their power-to-weight ratio in training.
- Bicycles have gears in different ratios to suit different conditions.
- The ratio of pupils to teachers in secondary schools is less than in primary schools.
- The scale factor of an enlargement is the ratio of corresponding lengths on the object and its image.
- When jogging at a steady 7 mph, the distance travelled is proportional to the time.
- The current I flowing in a circuit of fixed resistance R is proportional to the voltage applied V; that is, $V = IR$.
- The acceleration a of a mass m is proportional to the applied force F; that is, $F = ma$.
- On a hill of uniform slope, the height gained is proportional to the distance travelled along the slope. The gradient of the slope is the ratio of the increase in height to the horizontal distance.

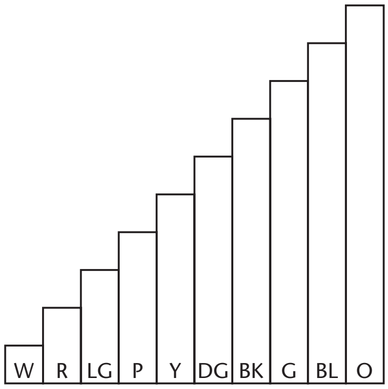
Resource 7b Introducing ratio and proportion

RATIO

Ratio is a way of comparing two numbers or quantities, by expressing how many times one number can be divided by the other.

The view of many pupils is that ratio is used only to compare parts of a whole. This is only one aspect of ratio. Ratio is not necessarily a part/part relationship.

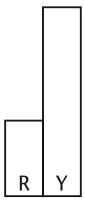
Assume that we have a set of coloured strips.



We will adopt shorthand for the lengths of the strips, using the symbol l_Y or l_R, for example, to represent the number of units of length of a strip:

1 unit	white (l_W)
2 units	red (l_R)
3 units	light green (l_{LG})
4 units	pink (l_P)
5 units	yellow (l_Y)
6 units	dark green (l_{DG})
7 units	black (l_{BK})
8 units	grey (l_G)
9 units	blue (l_{BL})
10 units	orange (l_O)

If we place two strips side by side, for example the red (2 unit) strip and the yellow (5 unit) strip, they would look like this:



We can compare the lengths of the two strips by finding a *difference*, or alternatively by finding a *ratio*.

ratio of l_Y to $l_R = 5:2$

$$\frac{l_Y}{l_R} = \frac{5}{2}$$

$$l_Y = \tfrac{5}{2} \times l_R \quad (\text{or } 2\tfrac{1}{2} \times l_R)$$

ratio of l_R to $l_Y = 2:5$

$$\frac{l_R}{l_Y} = \frac{2}{5}$$

$$l_R = \tfrac{2}{5} \times l_Y$$

Pick two different strips of your own choice and draw them below. Write out the ratios as in the example above.

Repeat with another two strips.

Another way of using coloured strips to illustrate ratio is to line up multiples ('trains') of each strip so that two coloured 'trains' are the same length, and then count the number of strips in each equal-length train. (Compare this with finding a common multiple.)

For example, the length of five red strips equals the length of two yellow strips, so the ratio $l_R : l_Y$ equals 2 : 5. (It is important to get the order correct; comparing the lengths of the two initial strips helps.)

When pupils are working with ratios they need to understand that:

- **Ratio involves division of one number by another.**
  - Sometimes the result will be a whole number.
  - Sometimes the result has to be left as a fraction or mixed number (or converted to an equivalent decimal or percentage form).
- **There are several ways of expressing a ratio:**
  - using colon notation;
  - using the fraction or division line (reading '/' as '÷');
  - using multiplication (reading '×' as 'of').
- **The inverse of the ratio *a* : *b* is *b* : *a*.**

  The associated fractions are $\frac{a}{b}$ and its reciprocal $\frac{b}{a}$.

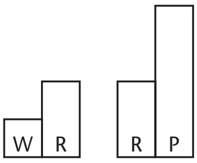
Other important points about the teaching of ratio and proportion are:

- At an appropriate stage, pupils should meet equivalent decimal and percentage forms of expressions, such as:
  $\frac{2}{5} = 0.4$, which is equivalent to 40%;
  $\frac{5}{2} = 2.5$, which is equivalent to 250%.
- Pupils' understanding is greatly assisted if they discuss different ways of expressing a ratio, rather than being introduced to different notations on completely separate occasions. This helps them to link ideas, use different forms and recognise equivalent expressions.

- The notion of an inverse operation is a key one in mathematics. In Key Stages 1 and 2, pupils are introduced to an operation and its inverse together – for example, addition with subtraction and multiplication with division. Understanding inverse ratios will help pupils later – for example, to calculate the original amount given the amount after a specified percentage change.

PROPORTION

A **proportion** is an equality of ratios.



For these two pairs of coloured strips, the ratios are equal. What other pairs of strips have the same ratio?

Working with the coloured strips, the complete proportional set for the ratios 1 : 2 and 2 : 1 is:

$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10}$ and $\frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{8}{4} = \frac{10}{5}$

Record all the other proportional sets you could make with the coloured strips.

Resource 7c Walt's lesson plan

Ratio and proportion: Year 7

ORAL AND MENTAL

Objectives

- Consolidate rapid recall of multiplication facts.
- Calculate mentally using ratio.

Activity

Remind class of concept of ratio, drawing on what they have done in previous two lessons.

Use counting stick to get pupils to follow the 4× and 10× multiplication tables simultaneously.

MAIN ACTIVITY

Objective

- Solve simple problems involving ratio.

Key vocabulary

Ratio, compare, 'for every', equivalent, lowest terms

Introduction

Remind class of previous activities involving ratio. Introduce exchange rates, leading to equivalent ratios and reducing to lowest terms. Ensure that pupils are clear on when they are using the same units. Use table from previous work on exchange rates to generate equivalent ratios.

Activity

Worksheet on ratio.

PLENARY

Show class picture by de Chirico and get pupils to spot the importance of shadows. Show shadow picture and discuss whether the shadows were observed at same time of day. Ask 'How can we tell?' 'What has ratio got to do with it?'

Homework

None

Resource 7d Observing Walt's lesson

Watch Walt working with his Year 7 top set. The lesson has three distinct parts: a starter, the main activities, and a final plenary.

Watch each part of the lesson separately. First, watch the starter, then consider and make notes on the questions below.

Oral and mental starter

- How did Walt make use of the OHP?
- What consideration did Walt give to the use of mathematical vocabulary?
- How did Walt make use of a counting stick?

Now watch the extract from the main part of the lesson. This shows Walt in discussion with two pairs of pupils who are working on proportion problems. When you have watched the extract, consider and make notes on the questions below.

Main activity

- How did pupils approach the ratio problems?
- What did the pupils record?

Now watch the final plenary, then consider and make notes on the questions below.

The plenary

- What strategies did Walt use to reinforce pupils' understanding of ratio and proportion?

Now that you have watched the whole lesson, think about it as a whole.

Reflections on the whole lesson

- Think about Walt's use of resources. Could you make use of similar resources with any of the classes that you teach? If so, make a note of them here.

- Were there any points about Walt's lesson that you would like to discuss with your head of department? If so, make a note of them here.

Resource 7e Proportions and graphs

Consider the set of fractions (or ratios) equivalent to 5:

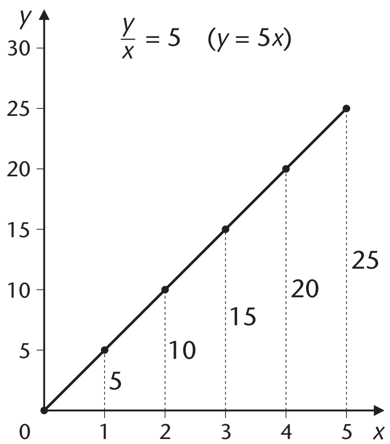
$$\frac{5}{1}=\frac{10}{2}=\frac{15}{3}=\frac{20}{4}=\frac{25}{5}=\ldots$$

The denominators {1, 2, 3, 4, 5, …} and numerators {5, 10, 15, 20, 25, …} are called **proportional sets** and the pairs of corresponding numbers (1, 5), (2, 10), (3, 15), (4, 20), (5, 25), … are said to be **in proportion**.

The relationship between these numbers can be illustrated graphically. Representing the set of denominators {1, 2, 3, 4, 5, …} by variable *x* and the set of numerators {5, 10, 15, 20, 25, …} by variable *y*, then:

$$\frac{y}{x}=5 \quad (\text{or } y = 5x)$$

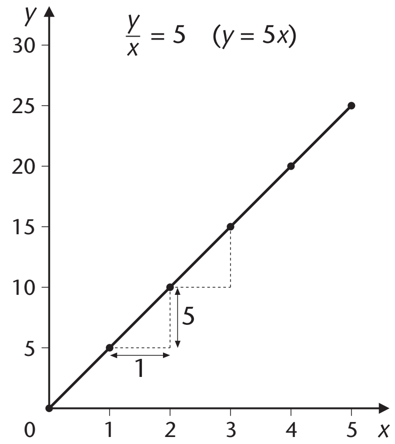
x	1	2	3	4	5	…
y	5	10	15	20	25	…



The graph illustrates the equal ratios $\frac{5}{1}, \frac{10}{2}, \frac{15}{3}, \ldots$, represented by the sides of right-angled triangles, enlarging from the origin. This visualisation of a proportion links with enlargement and similarity and with later applications in trigonometry, which will be considered in Module 8.

It is valuable for pupils to see the relationship $y = 5x$ expressed as $\frac{y}{x}=5$, since this draws attention to the equal ratios.

Another aspect to consider is how *y* changes with *x*. From the table of values above the graph, we can see that *y* increases by 5 for every 1 of *x*. By drawing a staircase pattern of triangles, as shown below, we can see that rises of 5 in *y* correspond to steps of 1 in *x*. This is an example of a constant **rate of change**.



Pupils should notice that the graphs of all proportions (equal ratios) are straight lines through the origin, with constant rate of change.

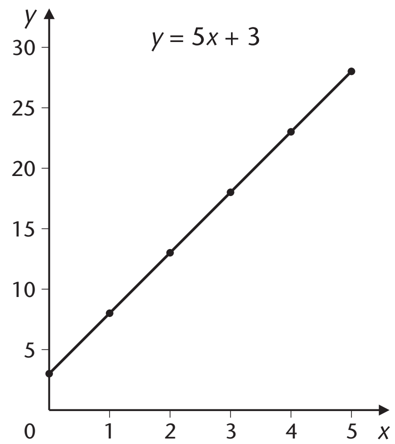
In its general form, the relationship $y = 5x$ is $y = mx$, where m is a constant. Functions of the form $y = mx$ are proportions, where m, which measures the **rate of change** or **gradient** of the graph, is called the **constant of proportionality**. (This type of relationship would be called **direct proportion** when it is necessary to distinguish it from **inverse proportion**.)

Depending on the quantities represented, the constant m may be thought of as a scale factor or multiplier (e.g. when dividing one distance or length by another) or as a rate, (e.g. when dividing distance by time, as in miles per hour).

Proportions are not the only relationships that involve a constant rate of change. Consider the function $y = 5x + 3$, and complete the following table of values:

x	1	2	3	4	5
y					
$\frac{y}{x}$					

The rate of change is still 5, but the ratios of successive pairs of values are not constant.



The relationship $y = 5x + 3$ is a **linear relationship**, since the graph is a straight line, but it does not go through the origin. The **general linear function** is of the form $y = mx + c$, where m is the rate of change of y with x. The constant m is a measure of the **gradient** of the graph and c is the **intercept** with the y-axis.

Resource 7f Proportion as a functional relationship

Consider the questions below and make notes on your responses.

What are the benefits to pupils if they are given opportunities to make the connections considered in this module?

How might an awareness of the connections help pupils when they are dealing with practical applications, whether in mathematics or other subjects?

Resource 7g Summary and further action on Module 7

Look back over the notes you have made during this module. Identify the most important things to consider and modify in your planning and teaching of ratio, proportion and proportional reasoning.

List two or three key points that you have learned.

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List two or three points to follow up in further study.

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List two or three modifications that you will make to your planning or teaching of ratio, proportion and proportional reasoning.

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List the most important points that you want to discuss with your head of department, or any further actions you will take as a result of completing this module.

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